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COMPUTER AND MATHEMATICAL SCIENCES LABORATORY PROJECT 5632
AIR FORCE CAMBRIDGE RESEARCH LABORATORIES, OFFICE OF AEROSPACE RESEARCH, UNITED STATES AIR FORCE, L.G. HANSCOM FIELD, MASS.

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Abstract

This paper presents a new method for minimizing a positive definite quadratic function subject to linear inequality constraints. It is based on a continuous deformation of the quadratic starting from one giving rise to an almost trivial problem and ending with the desired quadratic.

The method is particularly suitable for problems involving large numbers of inequality constraints, since the size of the variable space is independent of the number of inequalities.

The method is readily extended to linear programming by embedding the linear objective function in a second degree polynomial with positive definite quadratic component, solving the resulting problem by the above method, then driving the quadratic component to zero; that is, deforming the second degree programming problem back to the original linear one.

Contents

A Deformation Method for the Minimization of a Quadratic Function Subject to Linear Inequality Constraints

1. INTRODUCTION

In recent years there has been considerable research devoted to the problem of minimizing a positive definite quadratic function of n variables over a polyhedral region in Euclidean n space and a variety of methods have appeared in the literature. This paper presents a method closely related to the capacity methods of Houthakker[1] and Markowitz.[2] My reason for presenting another method, and perhaps adding to the confusion, is that their methods are tailored to a particular application and are less well suited to other applications. Furthermore, the present method is readily extended to linear programming, whereas the capacity methods do not seem capable of this extension.

This method, as well as those of Houthakker and Markowitz, involves manipulation of matrices whose sizes are independent of the number of linear inequalities defining the polyhedral region, in contradistinction to methods based on the simplex algorithm, such as those of Beale[3] and Frank and Wolfe,[4] in which the sizes of the matrices increase linearly with the number of inequalities. This is one of the principal reasons for considering algorithms alternative to ones based on the simplex algorithm.

Denote the polyhedral region by U, defined by the inequalities

$$\underline{a}_i{}'\underline{x} \geq b_i, \qquad i = 1, \ldots, p, \tag{1.1}$$

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where $\underline{x}' = (x_1, \ldots, x_n)$ is a vector variable, and $\underline{a}_i' = (a_{i1}, \ldots, a_{in})$, b_i, $i = 1, \ldots, p$ are a set of vector and scalar constants, respectively. The x_i's are not necessarily non-negative; however, if they are, these conditions are presumed included in (1. 1). Since any equality can be written as two inequalities, then (1. 1) also includes any equality conditions. The quadratic is given by:

$$Q(\underline{x}) = \sum_{j=1}^{n} \sum_{h=1}^{n} x_j r_{jk} x_k = \underline{x}' R\underline{x} \tag{1. 2}$$

where $R = \{r_{jk}\}$ is a positive definite n xn matrix. Since any minimization problem of the type under consideration is specified once the inequalities (1. 1) and the quadratic $Q(\underline{x})$ are given, a problem will be defined to be a pair (Q, U) where Q is a quadratic of the form Eq. (1. 2), and U is a polyhedral region. A solution to a problem will be a point in U minimizing the quadratic.

If the region U contains the origin, then this point is the solution; otherwise it is known that the solution is on the boundary of U and elementary calculus methods cannot be used to determine this point.

The general method used here, and by Houthakker and Markowitz, is to start with some point known to be in U and to formulate a problem whose solution is this particular point, then to continuously deform the problem into the original one while keeping track of the solution. Houthakker and Markowitz do this by means of an additional inequality $\underline{a}_{p+1}'\underline{x} \geq b_{p+1}$, where b_{p+1} is a parameter with the range $-\infty$ to ∞. This inequality bisects U into two parts: $U_1 = U \cap \{\underline{x} : \underline{a}_{p+1}'\underline{x} \geq b_{p+1}\}$ and $U_2 = U - U_1$ (one of these may be empty). For some value of b_{p+1}, U_1 will consist of a single point (or, in exceptional cases, a hyperplane). The minimum of $Q(\underline{x})$ over this particular U_1 is easily found. As b_{p+1} is then decreased (or increased), U_1 grows and the solution in U_1 moves continuously along the surface of U to the solution over U. Since U is a polyhedron, each point on its surface can be characterized as the intersection of a set of hyperplanes $\underline{a}_i'\underline{x} = b_i$, $i \in I$, where I is a subset of the integers $1, \ldots, p$. Thus, the solution can be kept track of by a set function $I(b_{p+1})$, which is a function of the parameter b_{p+1}, since the solution is a function of b_{p+1}. The accompanying algorithm is principally concerned with rules for determining when to add or delete indices from $I(b_{p+1})$.

Markowitz used this method in connection with what he called the portfolio problem, in which the equation $\underline{a}_{p+1}'\underline{x} = b_{p+1}$ has a special interpretation, and the solutions for all values of b_{p+1} are of interest. Therefore, the method seems quite well suited to the application.

In the method used here, one begins with a subproblem of the original problem, which is easy to solve, and obtained by setting $x_2 = \ldots = x_n = 0$ and replacing $Q(\underline{x})$

by the rank one quadratic $x_1^2 r_{11}$. Then one permits x_2 to be non-zero and deforms $x_1^2 r_{11}$ into the rank two quadratic $x_1^2 r_{11} + x_2^2 r_{22} + 2x_1 x_2 r_{12}$ in such a way that the solution moves continuously over U. The solution then becomes a function of t, and is determined explicitly. Thus its value, if known at the beginning of the deformation, will be known at the end. This is continued: the variables x_k are introduced one at a time; with each, the present quadratic $\sum_{i,j=1}^{k-1} x_i r_{ij} x_j$ is deformed into the quadratic $\sum_{i,j=1}^{k} x_i r_{ij} x_j$, one rank higher, until one returns to the original quadratic $Q(\underline{x})$. The deformation of the quadratic is achieved by replacing the elements r_{jk} of R by expressions tr_{jk} and $t + r_{kk}$ where t is a parameter taking values in an appropriate interval.

Along the way to the final solution the solutions to the subproblems are obtained. These solutions are needed in many applications. In particular, n, the number of variables, may itself be a variable and the minimum value of the quadratic a function of n. to be determined. For such problems our method seems the more natural one.

The addition of a linear term to the quadratic involves a minor modification of the algorithm (in principle, a suitable linear transformation will eliminate the linear part), and since a second degree polynomial can be continuously deformed into its linear component, any linear programming problem can be solved by embedding it in a second degree polynomial programming problem, solving the latter, and then deforming the second degree polynomial into its linear component.

The application which led to this method is the problem of linear, minimax prediction of deterministic processes using a quadratic loss function. In this application $x_1, \ldots, x_n$ are the coefficients of the linear predictor, while R is the covariance matrix of the observations. The observations have the form $y_j(\theta) + z_j$, $j = 1, \ldots, n$, where $y_j(\theta)$ is a discrete deterministic process indexed by a parameter θ, and z_j is a random variable.

Subject to certain restrictions, it can be shown that the problem of determining the x_i's reduces to a quadratic programming problem with, in general, an infinite number of linear inequality restrictions. In applications one can sometimes obtain a good approximation to the problem with a finite, though large, number of inequalities – hence the need for a quadratic programming algorithm which can handle large numbers of inequalities.

In this application it is generally desirable to obtain the predictors based on all sample sizes up to n, or to let n be a variable of the algorithm and stop according to some rule involving the expected loss.

The theory on which this algorithm is based is given in the next two sections. In Section 2 some definitions and preliminary facts stated as propositions are given. The algorithm is more directly based on three propositions given in Section 3. They are called propositions rather than theorems for lack of a more suitable word to describe some highly specialized results in the theory of linear spaces and convex functions.

The basic algorithm is described in Section 4, and an example given in Section 5; Sections 6, 7, and 8 present the treatment of three possible types of degeneracies which may occur in the basic algorithm; Section 9 treats computing procedures, while in Section 10 the method is extended to linear programming.

2. DEFINITIONS AND PRELIMINARIES

A problem, as previously stated in Section 1, is a pair (Q, U), where Q is a quadratic form and U is a polyhedral region in the space of the variables of the quadratic form. A solution to the problem is a point in U minimizing Q. It will be assumed that U is nonempty and does not contain the origin. Let I, with or without subscripts, be any subset of the integers $1, \ldots, p$ and let hI be the hyperplane defined by the equalities $\underline{a}_i{}'\underline{x} = b_i$, $i \in I$. Let t be a real parameter and $R(t) = \{r_{ij}(t)\}$ be an $n \times n$ matrix such that $r_{ij}(t)$ is either constant or linear in t, but not all $r_{ij}(t)$ are constant, and such that R(t) is positive definite for all t in its domain. This domain will be either a finite or infinite interval with the lower bound of this interval, α, always finite and the upper, β, finite or infinite. If the interval is finite, then $\alpha \leq t \leq \beta$; while if infinite, then $\alpha \leq t < \beta$. This interval is also denoted by T. For any polyhedral region U of $\underline{x}$ space, let $\underline{x}(t, U)$ denote the solution to the problem $(Q(\underline{x}, t), U)$ where $Q(\underline{x}, t) = \underline{x}'R(t)\underline{x}$. Thus, $\underline{x}(t, hI)$ is a solution to the problem $(Q(\underline{x}, t), hI)$ where hI is specified by $\underline{a}_i{}'\underline{x} = b_i$, $i \in I$.

Vectors will always be column vectors written in boldface (indicated by underlining). For any sets U, V, $U \subset V$ will denote U properly contained in V; $U \subseteq V$ will denote $U \subset V$ or $U = V$. To reduce the amount of notation, arguments of functions will be omitted where the functional dependence is clear from the text.

The polyhedron U is convex and so is any hyperplane hI. Moreover, $Q(\underline{x}, t)$ is a strictly convex function of $\underline{x}$, since $Q(\underline{x}, t)$ is positive definite. Therefore, $\underline{x}(t, U)$ exists and is unique for $t \in T$ and similarly for $\underline{x}(t, hI)$, if hI is nonempty.

For any $t \in T$ let I(t) be the set of all integers i, $1 \leq i \leq p$, such that $\underline{a}_i{}'\underline{x}(t, U) = b_i$. Then,

$$\underline{x}(t, hI(t)) = \underline{x}(t, U),^{*} \quad t \in T, \tag{2.1}$$

* Proof: $x(t, U)$ is in hI(t), and hI(t) is contained in any supporting hyperplane (that is, any hyperplane of the form $a'x = b$ such that $a'x > b$ for all $x \in U$, with equality holding for some $x \in U$) to U at the point $x(t, U)$. Now, since $x(t, U)$ minimizes $Q(x, t)$ over U, it follows that the ellipsoid consisting of all $\underline{x}$ such that $Q(x, t) = Q(\underline{x}(t, U), t)$ does not intersect U except at the point $\underline{x}(t, U)$. Since both U and this ellipsoid are convex,

so that an explicit formula for $\underline{x}(t, U)$ is given by the formula for $\underline{x}(t, hI(t))$. Note that $I(t)$ is maximal with respect to its given property.

The formula for $\underline{x}(t, hI(t))$ is obtained by means of the Lagrange multiplier method; that is, the unrestricted minimum of $Q(\underline{x}, t) - 2 \sum_k \lambda_k(\underline{a}_k{}'\underline{x} - b_k)$, $k \in I(t)$, with respect to $\underline{x}$ and $\underline{\lambda}$ is found. After differentiating, one obtains necessary and sufficient conditions for a minimum:

$$R(t)\underline{x} - A'\underline{\lambda} = \underline{0} \tag{2.2}$$

$$A\underline{x} = \underline{b}, \tag{2.3}$$

where A is the matrix with rows $\underline{a}_k{}'$, $k \in I(t)$. By assuming that A has full row rank; that is, the rows of A are linearly independent, the solution is:

$$\underline{x} = \underline{x}(t, hI(t)) = R^{-1}(t)\, A'\underline{\lambda} \tag{2.4}$$

$$\underline{\lambda} = \underline{\lambda}(t, hI(t)) = (AR^{-1}(t)\, A')^{-1}\, \underline{b}. \tag{2.5}$$

Direct use of this formula for computation would be very inefficient; a better method of computing $\underline{x}$ and $\underline{\lambda}$ is given in Section 9. However, it is satisfactory for expository purposes. If A is nonsingular, then (2.3) has a unique solution independent of $R(t)$. In this case $\underline{x}$ will not depend on t, while $\underline{\lambda}$ continues to do so (except if $\underline{b} = \underline{0}$, which is excluded by the hypothesis that U does not contain the origin).

For fixed A and b the coordinates of $\underline{x}$ and $\underline{\lambda}$ are rational in t. In this case a 0 value for the j'th coordinate of $\underline{\lambda}$ has the interpretation that for that particular value of t, the j'th condition, $\underline{a}_j{}'\underline{x} = b_j$, is redundant. More precisely,

> Let $f(\underline{x})$ be some function whose partial derivatives exist for all $\underline{x}$ and let $\hat{\underline{x}}' = (\hat{x}_1, \ldots, \hat{x}_n)$ be a stationary point of $f(\underline{x})$, subject to the k + r consistent and linearly independent linear restrictions, $\underline{a}_i{}'\underline{x} = b_i$, $i = 1, \ldots, k+r$. A necessary and sufficient condition that $\hat{\underline{x}}$ is also a stationary point of $f(\underline{x})$ subject to just the first k linear restrictions is that $\hat{\lambda}_{k+1} = \ldots = \hat{\lambda}_{k+r} = 0$, where $\hat{\lambda}_1, \ldots, \hat{\lambda}_{k+r}$, the Lagrange multipliers, are a solution to the equations (2.6)

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there is a separating hyperplane, H, (a hyperplane which is supporting hyperplane to both U and the ellipsoid), containing the point $\underline{x}(t, U)$ and hence, containing $hI(t)$. As the minimum of $Q(\underline{x}, t)$ over H is at $\underline{x}(t, H)$, the latter point also minimizes $Q(\underline{x}, t)$ over $hI(t)$.

$$\frac{\partial}{\partial x_j}\left(f(\hat{\underline{x}}) + \sum_{i=1}^{k+r} \lambda_i \, \underline{a}_i' \, \hat{\underline{x}}\right) = 0, \; j = 1, \ldots, n. ^* \tag{2.6}$$

For any I, the coordinates of $\underline{x}(t, hI)$ are rational in t, hence $\underline{x}(t, hI)$ is continuous. Furthermore, $\underline{x}(t, hI(t))$ is also continuous in t. This is true because $Q(\underline{x}, t)$ is continuous in $\underline{x}$ and t, hence $\min_{x \epsilon U} Q(\underline{x}, t) = Q(\underline{x}(t, hI(t)), t)$ is continuous in t. Since $\underline{x}(t, hI(t))$ is unique, it is, therefore, continuous.

The following section establishes the properties of I(t) on which the algorithm is based: as t traverses the interval T in either direction, I(t) makes only a finite number of changes in value and each change involves one or more (generally one) additions or deletions to the set I(t). At the points t where these additions or deletions occur, certain conditions hold which enable us to determine where these points are and which indices to add or delete from I(t).

3. THEORY OF THE ALGORITHM

For each $I \subsetneq \{1, \ldots, p\}$ and $j \notin I$ there are, at most, a finite number of points of intersection of $\underline{x}(t, hI)$ and $\{\underline{x}: \underline{a}_j' \underline{x} = b_j\}$ for $t \epsilon T$; hence, the total number of points of intersection for all pairs (I, j), $j \notin I \subsetneq \{1, \ldots, p\}$ is also finite. Let S^* be the set of all these points of intersection. Then at any point t^* where I(t) changes value, necessarily $\underline{x}^* = \underline{x}(t^*, hI(t^*)) \; \epsilon \; S^*$. Now it will be shown that to each $\underline{x}^* \epsilon S^*$ there correspond, at most, two points t of change of I(t), implying that I(t) changes value at no more than a finite number of points in T. If there are more than two points, then there exists $t_1 < t_2 < t_3$ such that $\underline{x}^* = \underline{x}(t_1, hI(t_1)) = \underline{x}(t_3, hI(t_3)) \neq \underline{x}(t_2, hI(t_2)) =$

---

*Proof: The sufficiency is immediate. For the necessity, if $\hat{\underline{x}}$ is a stationary point of $f(\underline{x})$ subject just to $\underline{a}_i' \underline{x} = b_i$, $i = 1, \ldots, k$, then there exist $\hat{\mu}_1, \ldots, \hat{\mu}_k$ satisfying

$$\frac{\partial}{\partial x_j}\left(f(\hat{\underline{x}}) + \sum_{i=1}^{k} \hat{\mu}_i \, \underline{a}_i' \, \underline{x}\right) = 0, \; j = 1, \ldots, n.$$

Subtracting the latter from the equation in (2.6) for $j=1, \ldots, n$, one obtains:

$$\sum_{i=1}^{k} (\hat{\lambda}_i - \hat{\mu}_i) \, \underline{a}_i + \sum_{i=k+1}^{r} \hat{\lambda}_i \, \underline{a}_i = \underline{0}.$$

Since the $\underline{a}_i$'s are linearly independent, $\hat{\lambda}_i = \hat{\mu}_i$, $i = 1, \ldots, k$, and $\hat{\lambda}_i = 0$, $i = k+1, \ldots, r$. Q. E. D.

$\underline{x}_2$. Since $Q(\underline{x},t)$ is strictly convex in $\underline{x}$ for fixed t, $Q(\underline{x}^*,t_1) < Q(\underline{x}_2,t_1)$, $Q(\underline{x}^*,t_3) < Q(\underline{x}_2,t_3)$, and $Q(\underline{x}_2,t_2) < Q(\underline{x}^*,t_2)$. On the other hand, as the elements of R(t) are linear in t, $Q(\underline{x},t)$ is linear in t for fixed $\underline{x}$. However, the last three inequalities cannot be true simultaneously, as the reader may readily verify with a sketch. Geometrically, this implies that $\underline{x}(t, hI(t))$ cannot describe a loop.

Call any point t, where I(t) changes value, a transition point and the change itself a transition. For definiteness take t as decreasing in $T = [\alpha, \beta]$. At a transition point t_j there are three possible types of transitions: (1) either one or more subscripts are added to I(t); (2) one or more are subtracted; (3) both subtraction and addition take place.

If Type (1), then $I(t_i+) \subset I(t_i) = I(t_i-)$; the alternative, $I(t_i+) = I(t_i) \subset I(t_i-)$ cannot occur because $\underline{x}(t_i, hI(t_i)) \in hI(t_i-)$ and, by definition, I(t) is always maximal. For a similar reason in Type (2), $I(t_i+) = I(t_i) \supset I(t_i-)$ and, in Type (3), $I(t_i+) \subset I(t_i) \supset I(t_i-)$. In the latter case, there are two transitions at a single transition point.

Let $I_1, \ldots, I_m$ be the sequence of values which I(t) takes, and $t_2, \ldots, t_{m-1}$ the sequence of transition points as t decreases in the interval $\beta > t > \alpha$. In order to have the index for the transitions coincide with the index for the transition points take $t_i = t_{i+1}$ if Type (3) above occurs at t_i. Noting that at t_j the transition $I_{i-1} \to I_i$ takes place, we have

$$
\begin{aligned}
I(t) &= I_i \quad \text{for } t_i > t > t_{i+1} \quad \text{if } t_i > t_{i+1}, \\
I(t_i) &= I_{i-1} \quad \text{if indices are added to I(t) at } t_i \text{ (Types (1) or (3))}. \qquad (3.1) \\
&= I_i \quad \text{if indices are subtracted from I(t) at } t_j \text{ (Type (2))}.
\end{aligned}
$$

It is convenient to extend the above results to the endpoints α and β. If $t_1 = \beta < \infty$, then define $I(t_1) = I_o$. In general, $I(t_1) = I(t_1-) = I_1$, so that I_o may equal I_1. Similarly, define $I(\alpha) = I(t_m) = I_{m+1}$, where I_m may equal I_{m+1}. If $\beta = \infty$ and $\lim_{t\to\infty} \underline{x}(t, hI_1) = \underline{x}(\infty, hI_1)$ exists, then $I(t_1)$ will also exist. It is assumed below that $\underline{x}(\infty, hI_1)$ and $I(t_1)$ exist where they are used. In the following section their existence is proved for the particular algorithm developed.

For use in the sequel note a few conclusions which follow from these definitions will now be given. There may not be more than two adjacent equal transition points and, on the other hand, no adjacent sets I_i, I_{i+1} which are equal. Slightly less elementary is

$$
\begin{aligned}
t_{i+1} &> t_{i+2} \quad \text{if } I_i \supset I_{i+1} \\
& \qquad\qquad\qquad\qquad\qquad (3.2) \\
t_i &> t_{i+1} \quad \text{if } I_i \subset I_{i+1}.
\end{aligned}
$$

Proof. If $I_i \supset I_{i+1}$, the $I(t_{i+1}) = I_{i+1}$ by (3.1). If $t_i = t_{i+1}$, then the top conclusion follows immediately. On the other hand, if $t_i > t_{i+1} = t_{i+2}$, then $I_i = I(t_{i+1}+) \subseteq I(t_{i+1}) = I_{i+1}$ which contradicts the hypothesis. The second statement of (3.2) is proved similarly.

A third conclusion from these definitions is

$$\underline{x}(t_i, hI(t_i)) \quad \epsilon\ hI_{i-1} \cap hI_i, \tag{3.3}$$

since t_i is the point of transition of I_{i-1} to I_i and $\underline{x}(t, hI(t))$, being continuous, must satisfy $\underline{a}_j{}'\underline{x}(t_i, hI(t_i)) = b_j$, $j \in I_{i-1} \cup I_i$. Hence, $\underline{x}(t_i, hI(t_i)) \in h(I_{i-1} \cup I_i) = hI_{i-1} \cap hI_i$.

These definitions and results are given in terms of decreasing t; for t increasing, obviously $t_i \leq t_{i+1}$, which affects (3.2) and a number of other statements. In addition $t+$ and $t-$ must be interchanged wherever they appear. Throughout this section t will continue to be taken as decreasing.

As t decreases, $\underline{x}(t, hI(t))$ describes a continuous path on the surface of U. If $I_i \supset I_{i+1}$ at the transition point t_{i+1}, then, geometrically, $\underline{x}(t, hI(t))$ leaves the hyperplane hI_i which may be regarded as an edge of U and moves in the higher dimensional hyperplane hI_{i+1} which may be regarded as a face relative to hI_i. By (2.6) it follows that at the point t_{i+1}, where the transition $I_i \rightarrow I_{i+1}$ occurs, the Lagrange multipliers $\lambda_j(t_{i+1}, hI_i)$ corresponding to the conditions $\underline{a}_j{}'\underline{x} = b_j$, $j \in I_i - I_{i+1}$, become zero. This is a necessary condition satisfied by the triple t_{i+1}, I_i, and I_{i+1}.

On the other hand, if $I_i \subset I_{i+1}$ at t_{i+1}, then $\underline{a}_j{}'\underline{x}(t_{i+1}, hI_i) - b_j = 0$ for $j \in I_{i+1} - I_i$, which is a necessary condition satisfied by the triple t_{i+1}, I_i, and I_{i+1} in this case.

Thus a necessary condition for a transition of either type at the point $t \succeq t_i$ is either

$$\lambda_j(t, hI_i) = 0 \quad \text{for some } j \in I_i \tag{3.4}$$

or

$$\underline{a}_j{}'x(t, hI_i) - b_j = 0 \quad \text{for some } j \notin I_i\,. \tag{3.5}$$

Neither of these conditions alone is sufficient for its respective type of transition. In particular, there may exist subsets J', J" such that $J' \subset I_i \subset J''$, and points t', $t'' < t_i$ such that the triple t', I_i, and J' satisfies (3.4) for all $j \in I_i - J$, while the triple t'', I_i, and J'' satisfies (3.5) for all $j \in J'' - I_i$. The transition point t_{i+1}, if it is one of the two points t', t'', by definition, must be the larger of the two (the smaller if t is increasing). Hence, any sufficient conditions must take account of both (3.4) and (3.5) (this explains the apparent circularity of Props. 1 and 2, below). Furthermore, the occurrence of (3.4) or (3.5) is not sufficient for a transition (of either type) since the multipliers can vanish at a non-transition point.

The following two propositions give sufficient conditions for the two types of transitions, $I_i \subset I_{i+1}$ and $I_i \supset I_{i+1}$, respectively. The conditions in the first are also necessary.

Proposition 1. Let J be a subset of the integers $1, \ldots, p$ such that $J \supset I_i$, $i = 1, \ldots, m-1$, and assume that t is a decreasing parameter. Necessary and sufficient conditions that a number t' and the subset J are t_{i+1} and I_{i+1}, respectively, are

(i) $t_i > t' \geq \alpha$

(ii) $\underline{a}_j{}'\underline{x}(t', hI_i) = b_j$, $j \in (J-I_i)$

(iii) J is the maximal subset satisfying (ii) at t'

(iv) t' is the largest number satisfying (i) - (iii) for some $J' \supset I_i$, where J' is a subset of $\{1, \ldots, p\}$

(v) The conditions (i), (iii), and (iv) of Prop. 2 are not satisfied for a number $t'' > t'$.

Proof: We first prove the necessity of these conditions, i.e., that $t' = t_{i+1}$ and $J = I_{i+1}$ satisfy (i) - (v) if $I_{i+1} \supset I_i$.

By definition, $t_i \geq t_{i+1}$. If $t_i = t_{i+1}$ then, again by definition, $t_{i+1} = t_{i+2}$ and, by (3.1), it follows that $I_i = I(t_i) = I(t_{i+1})$, while $I_{i+1} = I(t_{i+1}-)$. But $\underline{x}[t_{i+1}, hI(t_{i+1})] \in hI_{i+1} \cap hI_i \subset hI_{i+1}$ by (3.3); therefore, $I(t_{i+1}) \supseteq I_{i+1} \supset I_i = I(t_{i+1})$, implying that $I(t_{i+1}) = I_{i+1}$ and contradicting (3.2). Thus $t_i \neq t_{i+1}$, proving (i).

By definition of the transitions and transition points (ii), (iii), and (iv) are true. For (v), if there exists t'' and J', $t'' > t_{i+1}$, and $J' \subset I_i$, satisfying (i), (ii), and (iv) of Prop. 2, then $Q(\underline{x}(t, hJ'), t) \leq Q(\underline{x}(t, hI_i), t)$ in $t'' > t > t'' - \epsilon' \geq t_{i+1}$, for some $\epsilon' > 0$ by (iv) of Prop. 2, and, by (ii) of Prop. 2, equality cannot hold identically throughout the interval. But inequality contradicts the definition of I_i; hence, it follows that (v) is necessary.

Now, from (iii) and (iv) of the present proposition it follows that t_{i+1} and I_{i+1} are the unique pair satisfying (i) - (v); therefore, the conditions are also sufficient. Q. E. D.

Proposition 2. Assume that the restrictions $\underline{a}_j{}'\underline{x} = b_j$, $j \in I_i$, are linearly independent, and t is decreasing. If there exist t' and J, $J \subset I_i$, $0 \leq i \leq m-1$, such that (i) - (vii) below are satisfied, then $t' = t_{i+1}$ and $J = I_{i+1}$.

(i) $t_i \geq t' > \alpha$

(ii) $\lambda_j(t', hI_i) = 0$ for $j \in I_i - J$, and there exists at least one $j \in I_i - J$ such that $\lambda_j(t, hI_i) \neq 0$ in t

(iii) $\underline{a}_j{}'\underline{x}(t, hJ) - b_j \neq 0$ for $j \in I_i - J$

(iv) $\underline{x}(t, hJ) \in U$ for $t' \geq t \geq t' - \epsilon'$, for some $\epsilon' > 0$

(v) J is the unique smallest subset of I_i satisfying (i) - (iv)

(vi) t' is the largest number satisfying (i) - (iv) for some proper subset of I_i

(vii) conditions (i), (ii), and (iii) of Prop. 1 are not satisfied for some t'' $\geq$ t' and $J' \supset I_i$.

<u>Proof:</u> We first prove that $t' \geq t_{i+1}$, then the converse. Now, either $I_{i+1} \supset I_i$ or $I_{i+1} \subset I_i$. If the former, and $t_{i+1} > t'$, then I_{i+1} and t_{i+1} satisfy (i), (ii), and (iii) of Prop. 1, violating (vi) of this proposition; hence, $t' \geq t_{i+1}$ if $I_{i+1} \supset I_i$.

If, on the other hand, $I_{i+1} \subset I_i$, then it will be shown that t_{i+1} and I_{i+1} satisfy (i) - (v); hence, by (vi), it will follow that $t' \geq t_{i+1}$. By definition, (i) and (iii) are true while, by (3.2), $t_{i+1} > t_{i+2}$, so that t_{i+1} and I_{i+1} satisfy (iv), again by definition. By (2.6), $\lambda_j(t_{i+1}, hI_i) = 0$ for $j \in I_i - I_{i+1}$; moreover, the set of multipliers λ_j, $j \in I_i - I_{i+1}$, cannot all be identically 0 with respect to t for, if they are then, in an interval $t_{i+1} \geq t \geq t_{i+1} - \epsilon > t_{i+2}$, $\epsilon > 0$ (which exists by (3.2)) I_i rather than I_{i+1} is maximal, contradicting the assumption that t_{i+1} is a transition point. Thus, t_{i+1} and I_{i+1} satisfy (ii). Finally, if (v) is not true, then the strict convexity of Q(x, t) with respect to x' implies that $\underline{x}(t, hI_{i+1})$ does not minimize Q(x, t) in the interval $t_{i+1} > t > t_{i+1} - \epsilon'$. Thus, t_{i+1} and I_{i+1} satisfy (ii).

Therefore, under either assumption $I_{i+1} \subset I_i$ or $I_i \subset I_{i+1}$, $t' \geq t_{i+1}$ and, since these assumptions are mutually exhaustive, it follows that $t' \geq t_{i+1}$ without either assumption. If now, $t' > t_{i+1}$ then, since $\underline{x}(t, hI_i)$ is the unique minimum of $Q(\underline{x}, t)$ over U in the interval $t' > t > t_{i+1}$ and, since $J \subset I_i$ implies $Q(\underline{x}(t, hJ), t) \leq Q(\underline{x}(t, hI_i), t)$ in this interval, then it follows from (iv) that $\underline{x}(t, hJ) = \underline{x}(t, hI_i)$ in a subinterval of this interval. But, since $\lambda(t, hI_i)$ is rational in t, it follows by Eq. (2.6) that $\lambda_j(t, hI_i) \equiv 0$ for $j \in I_i - J$, contradicting (ii). Therefore, $t' \leq t_{i+1}$ and, together with the previous result, it follows that $t' = t_{i+1}$.

It remains to show that $J = I_{i+1}$. First we prove that $I_{i+1} \subset I_i$. If $t_{i+1} = t' = t_i$, then condition (i) of Prop. 1 is violated; hence, $I_{i+1} \supset I_i$. The only alternative is $I_{i+1} \subset I_i$. On the other hand, if $t_i > t_{i+1}$ then (i) of Prop. 1 is satisfied. Now we show that if $I_{i+1} \supset I_i$, then (ii) and (iii) are also satisfied, thus violating (vii) of the present proposition. Thus the conclusion again will be $I_{i+1} \subset I_i$. By (3. 3) $\underline{x}(t_{i+1}, hI(t_{i+1})) \in hI_i \cap hI_{i+1}$. Therefore, since $I(t_{i+1})$ is maximal, $I(t_{i+1}) \supseteq I_i \cup I_{i+1}$. But, by (3. 1), $I(t_{i+1})$ equals either I_i or I_{i+1}, hence, $I(t_{i+1}) = I_i \cup I_{i+1}$. Therefore, if $I_{i+1} \supset I_i$, then $I(t_{i+1}) = I_{i+1}$, and t_{i+1} and I_{i+1} satisfy (ii) and (iii) of Prop. 1.

Thus $I_{i+1} \subset I_i$. Now if $I_{i+1} \supset J$, then (iv) is violated while, since (i) - (iv) are necessary for I_{i+1}, if $I_{i+1} \not\supset J$, then (v) is violated. Therefore, $I_{i+1} = J$. Q. E. D.

The conditions of Prop. 2 are only sufficient because it is possible for t_{i+1} and I_{i+1} to satisfy all the conditions except (v). This is illustrated in Figure 1, for which the following assumptions are made: hI_{i+1} coincides with an edge of U to the right of the vertex $\underline{x}(t_{i+1}, hI_i)$; $\underline{x}(t_{i+1}, hI_i) = \underline{x}(t_{i+1}, hJ)$; $\underline{x}(t, hI_{i+1})$ moves along hI_{i+1} in the direction of the arrow as t decreases; $\underline{x}(t, hJ)$ moves along the dashed line in the direction of the arrow as t decreases, but the dashed line is outside U. Then I_{i+1} will satisfy all the conditions except (v) of Prop. 2 while J will satisfy (v), but not (iv). This contingency will be called a degeneracy of Type 3, one of three possible types which are treated in a similar manner; namely, by detouring around the region in $\underline{x}$ space where the difficulty occurs, in this case around the point $\underline{x}(t_{i+1}, hI_i)$. This degeneracy is treated in Section 8.

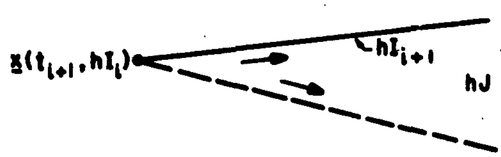

Figure 1.

Proposition 2 excludes the case where the linear restrictions corresponding to I_i are linearly dependent. It is very desirable, for computational reasons, that the restrictions be linearly independent. The contingency which arises when, for some transition point t_i, the rows $\underline{a}_j'$, $j \in I(t_i)$, are linearly dependent is called a degeneracy of Type 2 which is treated in Section 7.

If $t' = t_i$ in Prop. 2 then there results a double transition at the point t_i. This may occur in two ways, although they are both generally rare events. Figures 2 and 3 illustrate these double transitions. The straight lines and surfaces cut off by these lines indicate hyperplanes (the lines being the intersections of the surfaces), while the trajectory of $\underline{x}(t, U)$ is indicated by the straight or curved lines with arrows.

In Figure 2 $\underline{x}$ is tangent to a hyperplane at the point t_i. This tangency is equivalent to $\underline{a}_j{}'\underline{x}(t, hI_i) - b_j = 0$ having a double root at $t_i = t_{i+1}$ for $j \in I_{i+1} - I_i$. In Figure 3 $\underline{x}$ undergoes an abrupt change of direction at t_i, touching the hyperplane hI_{i+1} for just the single parameter point $t_i = t_{i+1}$. This event is equivalent to $\underline{a}_j{}'\underline{x}(t_{i+1}, hI_i) - b_j = 0$, $j \in I_{i+1} - I_i$, and $\lambda_k(t_{i+1}, hI_i) = 0$ for one or more $k \in I_i$. If neither of the two events illustrated by the figures occurs, it can be assumed that $t_i > t_{i+1}$.

Condition (iii) is added to Prop. 2 only because the I_i have been defined as maximal. From the proof of (2.6) it follows that if one first finds J ignoring (iii), then the addition of any subscripts to J in order to satisfy (iii) will not affect the λ's nor, of course, $\underline{x}$.

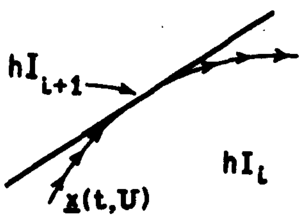

Figure 2

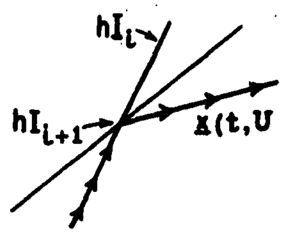

Figure 3

4. THE GENERAL FORM OF THE ALGORITHM

Recall that a problem was defined to be a pair $(Q(\underline{x}), U)$, where $Q(\underline{x})$ is a quadratic form in $\underline{x}$, and U is a polyhedral region in $\underline{x}$ space. The algorithm consists of n stages; in the k'th stage, $1 \leq k \leq n$, the original problem is solved subject to the additional restrictions $x_{k+1} = \ldots = x_n = 0$ (since these can be written as $2(n - k)$ inequalities, a new polyhedral region is obtained, hence a new problem). The stages are linked together by having the solution to each stage as the initial solution to the succeeding stage. To describe this more precisely, it is necessary to first introduce some terminology.

If $(Q(\underline{x}, t), U)$, $\alpha \leq t \leq \beta$, is a family of problems with parameter t, t taken as decreasing, then the initial and terminal solutions relative to this family of problems are defined to be $\underline{x}(\beta, U)$ and $\underline{x}(\alpha, U)$. Call $I(\beta) = I_o$ the initial restriction set, and $I(\alpha) = I_m$ the terminal restriction set relative to the family. If t is taken as increasing from α to β, then the designations 'initial' and 'terminal' are reversed. For brevity the designation 'problem' for 'family of problems' will be used.

Let ${}_1R_k(t)$ equal

$$\begin{bmatrix} r_{11} & \cdots\cdots & r_{1,k-1} & 0 \\ \cdots & \cdots & \cdots & \cdots \\ r_{k-1,1} & \cdots\cdots & r_{k-1,k-1} & 0 \\ 0 & \cdots\cdots & 0 & r_{kk}+t \end{bmatrix} \tag{4.1}$$

and ${}_2R_k(t)$ equal

$$\begin{bmatrix} r_{11} & \cdots\cdots & r_{1,k-1} & tr_{1k} \\ \cdots & \cdots & \cdots & \cdots \\ r_{k-1,1} & \cdots\cdots & r_{k-1,k-1} & tr_{j-1,k} \\ tr_{k1} & \cdots\cdots & tr_{k,k-1} & r_{kk} \end{bmatrix} . \tag{4.1.1}$$

Note that in (4. 1) only the lower right element depends on t, while in (4. 1. 1) t multiplies every element in the right column and lower row except r_{kk}.

The k-1'st stage is the problem of minimizing $\underline{x}'\,{}_1R_{k-1}(1)\underline{x}$, where $\underline{x}' = (x_1, \ldots, x_{k-1})$, subject to

$$\sum_{j=1}^{k-1} a_{ij}x_j \geq b_i, \quad i = 1, \ldots, p, \qquad x_k = \ldots = x_n = 0. \tag{4.2}$$

Assuming this problem has been solved, the elements of interest for the k'th stage, $k = 2, \ldots, n$, are the terminal restriction set $I(1)$ and the terminal solution $\underline{x}(1, hI(1))$ which lies on the hyperplane $hI(1)$. The k'th stage is subdived into two subproblems. The matrix of the quadratic of the first subproblem of the k'th stage is ${}_1R_k(t)$ with t going from ∞ to 0, and the polyhedral region is defined by

$$\sum_{j=1}^{k} a_{ij}x_j \geq b_i, \quad i = 1, \ldots, p \qquad x_{k+1} = \ldots = x_n = 0. \tag{4.3}$$

It is shown below that the terminal solution and restriction set of the k-1'st stage equal, respectively, the initial solution and initial restriction set of the first subproblem of the k'th stage. It follows that the solution is continuous between stages.

At the conclusion of the first subproblem the term $x_k^2 r_{kk}$ has been added to the quadratic. The second subproblem adds the cross product terms by taking ${}_2R_k(t)$ as the matrix of the quadratic of this subproblem, and allowing t to go from 0 to 1, unlike the first subproblem where t went from ∞ to 0. The conditions in (4.3) are maintained for both subproblems of the k'th stage. It is shown below that terminal and initial solutions and restriction sets of adjacent subproblems within a given stage are identical.

Our theory makes use of the strict convexity of the quadratic $Q(\underline{x}, t)$, with respect to $\underline{x}$, for all t in the range of interest, which is equivalent to the positive definiteness of $R(t)$. By hypothesis, ${}_2R_k(1)$ and ${}_1R_k(t)$, $t \geq 0$, are positive definite for $k = 1, \ldots, n$, while from ${}_2R_k(t) = (1-t)\,{}_1R_k(0) + t\,{}_2R_k(1)$, it follows that ${}_2R_k(t)$ is positive definite for $0 \leq t \leq 1$, $k = 2, \ldots, n$.

The identity of solution and restriction sets between adjacent subproblems and stages is important because the propositions of Section 3 only provide the basis of rules for adding or deleting elements to the set I(t); hence, for each subproblem and stage, it is essential that we have an initial restriction set. In the first stage an initial restriction set is unnecessary because the problem is transparent; a solution can be found almost at sight, provided a solution exists. Non-existence of a solution is one of the three types of degeneracy and is treated in Section 6.

In each of the subproblems into which the k'th stage is divided we make use of Props. 1 and 2 to traverse the associated range of t in a finite series of jumps from one transition point to the next. The method can be explained most clearly by an example given in the following section.

Non-degeneracy is assumed in this example. Recall that the three possible types of degeneracy are: (1) failure of a solution to exist in the first stage, which is equivalent to the non-existence of any $\underline{x}$ satisfying (4.3) for $k = 1$; (2) the matrix A in (2.2) - (2.5) does not have full rank, so that $\underline{x}$ and λ as defined by (2.4) and (2.5) fail to exist; (3) a transition of the type $I_{i+1} \subset I_i$ which fails to satisfy condition (iv) or (v) of Prop. 2, as explained in Section 3.

The subdivision of the over-all problem into stages, and each stage into subproblems, creates a chain of subproblems, ordered by the sequence in which they are solved. This section is concluded with a proof of the identity of terminal and initial solution and restriction sets of adjacent subproblems in this chain.

<u>Proposition 3.</u> Assuming non-degeneracy, the terminal solution and terminal restriction set of any subproblem is identical, respectively, to the initial solution and initial restriction set of the following subproblem.

<u>Proof:</u> The proof for the second subproblem of the k-1'st stage and the first subproblem of the k'th stage, $k = 2, \ldots, n$, will be given first

Throughout this paper it is assumed that ${}_2R_n(1)$ is positive definite, hence so is ${}_1R_k(t)$ for $0 \leq t \leq \infty$, $k = 2, \ldots, n$. Therefore, because A, in (2.2) and (2.3), is of full rank by our non-degeneracy assumption, it follows that, for all t, $0 \leq t \leq \infty$, there exists a solution $\underline{x}(t) = (x_1(t), \ldots, x_k(t))$, given by (2.4), to the problem determined by $\underline{x}'\,{}_1R_k(t)\underline{x}$ and the inequalities (4.3). Then, for any fixed vector $\underline{y}' = (y_1, \ldots, y_{k-1}, 0)$ satisfying Eq. (4.3), this solution must satisfy $\underline{x}'(t)\,{}_1R_k(t)\underline{x}(t) \leq y'_1\, R_k(t)_{\underline{y}} = \text{constant} < \infty$. Hence, the term $x_k^2(t)\,(r_{kk} + t)$ in $\underline{x}'(t)\,{}_1R_k(t)\,\underline{x}(t)$ must remain bounded as $t \to \infty$, implying $|x_k(t)| \to 0$ as $t \to \infty$. Letting $\underline{x}(\infty)$ be any limiting point of $\underline{x}(t)$ as $t \to \infty$ (at least one limiting point exists since $\underline{x}(t)$ lies in some bounded subset) it follows that $\underline{x}(\infty)$ satisfies Eq. (4.2) and minimizes

$$\sum_{j=1}^{k-1} \sum_{j'=1}^{k-1} x_j r_{jj'} x_{j'} \; . \tag{4.4}$$

Any solution to the problem defined by (4.2) and (4.4) is a terminal solution for the k-1'st stage. Furthermore, since (4.4) is strictly convex with respect to $(x_1, \ldots, x_k)$, $\underline{x}(\infty)$ is unique.

It follows, immediately, that $I(\infty) = I_0$ exists and equals the terminal restriction set of the k-1'st stage.

The corresponding result for the pair of subproblems within a stage follows immediately from the fact that ${}_1R_k(0) = {}_2R_k(0)$. Q. E. D.

5. EXAMPLE 1: The algorithm when no degeneracies are present

The purpose of this example, and the other examples below, is to exhibit how the algorithm proceeds; therefore, to avoid distractions from the main ideas no special computational techniques are used for the solution of the equations which arise. In Section 9 some computational techniques are given which, it is believed, make even large scale problems readily accessible to present day automatic computers.

For the first example

$$Q(\underline{x}) = 3x_1^2 + 2x_2^2 + 2x_1x_2 \tag{5.1}$$

is the quadratic, and U is the region defined by $\underline{a}_j'\underline{x} \geq b_j$, $j = 1, \ldots, 6$, where $\underline{a}_j$, b_j are given in the following table:

j	a_{j1}	a_{j2}	b_j
1	1	2	4
2	1	1	3
3	3	1	6
4	1	-1	- 2
5	-1	-2	-10
6	-1	4	- 5

The region U is illustrated in Figure 4.

For the first stage let $Q(x_1) = 3x_1^2$ and set $x_2 = 0$.

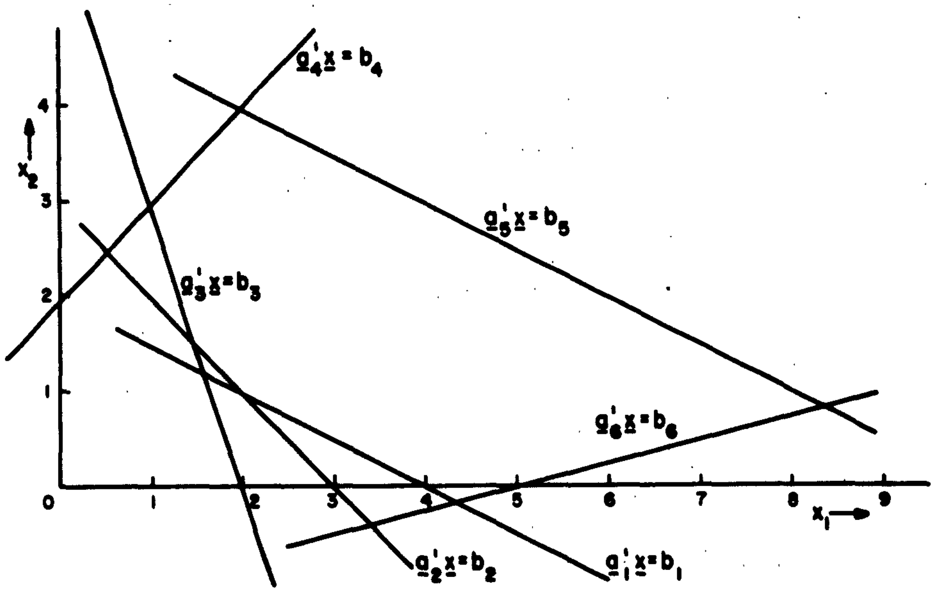

Figure 4.

As is seen from Figure 4, the section $U_1 \subset U$ to which the solution is restricted consists of all points $(x_1, 0)$ with $4 \leq x_1 \leq 5$. Thus the solution is $x_1 = 4$, $x_2 = 0$. This is the terminal solution to the first stage; as it lies on the line $\underline{a}_1{}'\underline{x} = b_1$, $\{1\}$ is our terminal restriction set.

The first subproblem of the second stage is to minimize

$$Q(\underline{x}, t) = 3x_1^2 + (2+t)\, x_2^2 \tag{5.2}$$

over U, with t going from ∞ to 0 and with $\{1\} = I_o$, the initial restriction set. The Lagrangian is, therefore,

$$3x_1^{\,2} + (2+t)x_2^{\,2} - 2\lambda_1(x_1 + 2x_2)$$

from which we obtain the conditions

$$3x_1 - \lambda_1 = 0$$

$$(2+t)\,x_2 - 2\lambda_1 = 0$$

holding at the minimum. Together with the restriction, $x_1 + 2x_2 = 4$, this system of equations has the solution

$$x_1 = \frac{4\,t + 8}{t + 14}, \quad x_2 = \frac{24}{t + 14}, \quad \lambda_1 = 3\,x_1 .$$

As a check note that, as $t \to \infty$, $x_1 \to 4$ and $x_2 \to 0$, as required.

Conditions (3.4) and (3.5) are now used to determine the first transition point. Condition (3.4) is not satisfied for $t \geq 0$. Condition (3.5) is $(4t+8)\,a_{j1} + 24a_{j2} - b_j(t+14) = 0$. To find the values of t satisfying this for $j \neq 1$ one solves for t, and obtains

$$t = \frac{14b_j - 8a_{j1} - 24\,a_{j2}}{-b_j + 4\,a_{j\,1}} . \tag{5.3}$$

Because of condition (iv) of Prop. 1 it is desirable to consider the largest value of (5.3) as the first candidate for a transition; this is $t = 10$ for $j = 2$.

As the conditions of Prop. 1 are satisfied for $t = 10$ and $J = \{1, 2\}$ they are designated t_2 and I_2 respectively (note that $I_o = I_1$ in this case). Carrying out the minimization of $Q(\underline{x}, t)$ on the new hyperplane hI_2 results in

$$x_1 = 2,\; x_2 = 1,\; \lambda_1 = t - 4,\; \lambda_2 = -t + 10,$$

where, of course, x_1 and x_2 are independent of t since hI_2 is a point. Condition (3.5) is inoperative since x_1 and x_2 do not depend on t. From (3.4) $\lambda_1 =$ 0 when $t = 10$. Of these two values only $t = 4$ is a candidate for the next transition point since, by the remarks following Prop. 2, t_2 is not a double transition point. On the other hand, the result $t = 10$ serves as a check on the computations since

$t = 10$ is precisely the value of t for which the condition $\underline{a}_2'\underline{x} = b_2$ should become redundant.

Checking the conditions of Prop. 2 for $t = 4$ it is seen that all the conditions are satisfied except possibly (iv), which cannot be checked at this point in the computations. Therefore, assume that (iv) holds and continue, reserving the check for later. Tentatively, $t_3 = 4$, $I_3 = \{2\}$. On hI_3,

$$x_1 = \frac{3t+6}{t+5}, \quad x_2 = \frac{9}{t+5}, \quad \lambda_2 = 3x_1,$$

and one can now check that (iv) of Prop. 2 is satisfied, for

$$(3t+6)a_{11} + 9a_{12} - (t+5)\,b_1 \geq 0, \qquad (5.4)$$

for $t = 4-$. If (iv) is not satisfied then a degeneracy of Type 3 has occurred at $t = 4$ requiring one to backtrack to deal with this degeneracy.

The procedure of using (3.4) and (3.5) to find a candidate for the next transition point is repeated. (3.4) fails to be satisfied for $t \geq 0$, while, from (3.5),

$$t = \frac{5b_j - 6a_{j1} - 9a_{j2}}{-b_j + 3a_{j1}},$$

which has maximum value 1 for $j = 3$. By Prop. 1 it follows that $t_4 = 1$ and $I_4 = \{2, 3\}$.

On hI_4

$$x_1 = \frac{3}{2} = x_2, \quad \lambda_2 = \frac{9+9t}{4}, \quad \lambda_3 = \frac{3-3t}{4}.$$

As neither (3.4) nor (3.5) is satisfied for $t \geq 0$ it follows that there are no more transition points as $t \to 0$; hence, at $t = 0$, $x_1 = x_2 = 3/2$ is the terminal solution and $\{2, 3\}$ is the terminal restriction set. This completes the first subproblem of the second stage.

For the second subproblem take

$$Q(\underline{x}, t) = 3x_1^2 + 2x_2^2 + 2tx_1x_2, \qquad (5.5)$$

and the initial restriction set $\{2, 3\}$. This time t goes from 0 to 1. We obtain

$$x_1 = x_2 = \frac{3}{2}, \quad \lambda_2 = \frac{9+6t}{4}, \quad \lambda_3 = \frac{3}{4}$$

and, as neither (3.4) nor (3.5) is satisfied for $0 \le t \le 1$, one concludes that no transitions take place as $t \to 1$. The final result is, therefore, $x_1 = x_2 = 3/2$.

This simple example exhibits all the steps of the algorithm when no degeneracies occur. Note that it is not necessary to vary t continuously, even though we spoke of doing this in the theory; instead, the transition points are found as the roots of equations linear in t. In general, for larger problems, second degree polynomials in t arise after the second stage, but the roots are still easily found by the well known formula.

The following section describes the method of treating the degeneracy occasioned by the lack of a solution to the first stage.

6. DEGENERACY OF TYPE 1: No solution to the first stage

Assume that the first stage has no solution; that is, there is no x_1 satisfying $a_{j1}x_1 \ge b_j$, $j = 1, \ldots, p$. To start the algorithm modify the original problem by adding the term x_o^2 to the quadratic and x_o to the left side of each of the inequalities, giving a new quadratic, $x_o^2 + \underline{x}'R\underline{x}$, and a new region defined by

$$x_o + \underline{a}_j'\underline{x} \ge b_j, \quad j = 1, \ldots, p. \tag{6.1}$$

The first stage of this new problem has the quadratic x_o^2 and the additional conditions $x_1 = x_2 = 0$. This stage has a solution for there certainly exists an x_o satisfying (6.1), viz., $\max_j b_j$, which also minimizes x_o^2 among all x_o satisfying the restraints.

Therefore, except for the introduction of the new variable x_o, the algorithm is the same as in Example 1. At the end, though, x_o will, in general, have a non-zero value, and to obtain the solution to the original problem x_o must be driven to zero. This is done by replacing x_o^2 in the quadratic by $(t+1)x_o^2$ and letting t go from 0 to ∞. This procedure is exactly the reverse of the procedure used in the first subproblem of any stage, where a variable which up to then had been fixed at 0 was allowed to become non-zero. The procedure is exhibited in the following example.

Example 2: Let

$$Q(\underline{x}) = 2x_1^2 + 3x_2^2$$
$$\underline{a}_j'\underline{x} \ge b_j, \quad j = 1, \ldots 5 \tag{6.2}$$

where

i	a_{j1}	a_{j2}	b_j
1	1	2	4
2	3	1	6
3	1	-1	-2
4	-1	-2	-10
5	-1	4	-4

The polygon defined by the inequalities lies entirely in the positive quadrant, so that no solution of the form $(x_1, 0)$ or $(0, x_2)$ exists. The modified problem is

$$Q(\underline{x}) = x_o^2 + 2x_1^2 + 3x_2^2, \tag{6.3}$$

$$x_o + a_{j1}x_1 + a_{j2}x_2 \geq b_j, \quad j = 1, \ldots 5, \tag{6.4}$$

where $\underline{x}' = (x_o, x_1, x_2)$.

For the first stage, upon setting $x_1 = x_2 = 0$, the solution $x_o = \max_j b_j = b_2 = 6$, and the terminal restriction set $\{2\}$ are obtained.

The procedure is continued just as in Example 1 obtaining, at the end of the third stage, the solution $x_o = x_1 = 4/3$, $x_2 = 2/3$, and the terminal restriction set $\{1, 2\}$. Therefore, to drive x_o to zero, take

$$Q(\underline{x}, t) = (t+1)\, x_o^2 + 2x_1^2 + 3x_2^2 \tag{6.5}$$

and $I_o = \{1, 2\}$, and obtain

$$x_o = \frac{52}{14 + 25t}, \quad x_1 = \frac{12 + 40t}{14 + 25t}, \quad x_2 = \frac{-4 + 30t}{14 + 25t}$$

$$\lambda_1 = \frac{38t - 12}{14 + 25t}, \quad \lambda_2 = \frac{12 + 14t}{14 + 25t}.$$

Condition (3.4) is not satisfied by λ_1 or λ_2 for $t \geq 0$, while from (3.5)

$$t = \frac{14b_j - 12a_{j1} + 4a_{j2} - 52}{-25b_j + 40a_{j1} + 30a_{j2}}$$

which is negative for $j = 3, 4, 5$. Thus, the necessary condition for a transition is not satisfied for $t \geq 0$. Letting $t \to \infty$ gives, in the limit, $x_o = 0$, $x_1 = 8/5$, $x_2 = 6/5$, where the latter two are the solution to the original problem (6. 2).

7. DEGENERACY 2: A not of full rank

The matrix A is a variable matrix depending on I(t); i. e., A has rows $\underline{a}_j'$, $j \in I(t)$. A Type 2 degeneracy occurs when, for some particular restriction set I_i, A does not have full row rank, so that $\underline{\lambda}(t, hI_i) = (AR^{-1}(t)A')^{-1}\underline{b}$ and $\underline{x}(t, hI_i) = R^{-1}(t)\, A'\underline{\lambda}$ fail to exist. This degeneracy will announce itself by the 'blowing up' of the computation of $\underline{\lambda}(t, hI_i)$ and $\underline{x}(t, hI_i)$; i. e., at some point in the computation one will be attempting to divide by zero.

Assume that i is the first index for which this degeneracy occurs and that $i \geq 2$; the case $i = 1$ is treated slightly differently. Assume that t is decreasing and let $\epsilon > 0$ satisfy $t_{i-1} > t_i + \epsilon$ if $t_{i-1} > t_i$ or $t_{i-2} > t_i + \epsilon$ if $t_{i-1} = t_i$; according to the definition of a transition point, these are the only alternatives. Both cases are treated alike, but for definiteness assume $t_{i-1} > t_i$. t is fixed at the value $t_i + \epsilon$, so that $I(t_i + \epsilon) = I_{i-1}$; and now, since i is the first index for which Type 2 degeneracy occurs, it follows that A has full row rank. Next introduce the new variable x_o and new parameter s as in the first degeneracy, except that in adding x_o to the restrictions, its accompanying coefficients must be such that A has full row rank. For this purpose it is desirable to have available rules for generating linearly-independent columns, as, for example, the columns of the identity matrix.

In case $i = 1$ and $\beta = \infty$, i.e., the first subproblem of a stage, one cannot fix t at $t_1 + \epsilon$; it is necessary, instead, to go back to the previous subproblem and introduce x_o and s before t reaches zero. This is the only way the case $i = 1$ may differ from the case $i > 1$.

If t is increasing, fix t at $t_i - \epsilon$, $\epsilon > 0$, but otherwise the procedure is the same.

Example 3: To the inequalities of Example 1 add a seventh, $5x_1 + 7x_2 \geq 17$. In the first subproblem of the second stage, when $t = t_2 = 10$, $5x_1 + 7x_2 = 17$, since $x_1 = 2$, $x_2 = 1$ at t_2. In this case $I_2 = \{1, 2, 7\}$, and, having 3 elements, degeneracy results. Fixing t at 11 and introduce x_o and s, the new problem is

$$Q(x_o, \underline{x}, s) = (1+s)x_o^2 + 3x_1^2 + (2+t)x_2^2, \quad t = 11$$

$$x_o + \underline{a}_j'\underline{x} \geq b_j, \qquad j = 1, \ldots, 7$$

with the restriction set $\{1\}$. The solution is

$$x_0 = \frac{156}{64+25s}, \quad x_1 = \frac{52(1+s)}{64+25s}, \quad x_2 = \frac{24(1+s)}{64+25s}, \quad \lambda_1 = \frac{156(1+s)}{64+25s}. \tag{7.1}$$

The largest s satisfying either (3. 4) or (3. 5) is 168. Therefore, fixing s at 179, t is allowed to resume the role of parameter. The restriction set I_1 is now $\{1\}$, and the successive results are $t_2 = 10.934$, $I_2 = \{1, 7\}$; $t_3 = 6.52$, $I_3 = \{7\}$; $t_4 = 6.208$, $I_4 = \{2, 7\}$; $t_5 = 3.124$, $I_5 = \{2\}$; $t_6 = 1.051$, $I_6 = \{2, 3\}$; and finally $t_7 = 0$, $I_7 = \{2, 3\}$. At this point s is driven back to ∞ (though it could have been done earlier) by fixing t at 0 and letting s be the parameter with range $179 \leq s \leq \infty$, and with $I_1 = \{2, 3\}$. There are no transitions as $s \to \infty$, and, in the limit, $x_1 = x_2 = 3/2$ is obtained, as in Example 1.

8. DEGENERACY 3: Condition IV of Proposition 2 not satisfied

The degeneracy announces itself by the failure of the candidate J to satisfy (iv) or (v). This degeneracy is resolved in the same way as in the previous degeneracy.

9. COMPUTATION OF $\underline{x}$ (t, h I_1) and $\underline{\lambda}$ (t, h I_1)

In the simple examples given above $\underline{x}(t, hI_1)$ and $\underline{\lambda}(t, hI_1)$ were obtained as explicit function of t by the straightforward solution of equations. While this is easy to do for a few variables, it becomes very time consuming for larger numbers of variables because of the presence of the parameter t, which precludes the use of ordinary numerical methods. Fortunately, however, a relatively simple formula for $\underline{x}(t, hI_1)$ and $\underline{\lambda}(t, hI_1)$ can be obtained.

${}_jR_k(t)$, j = 1, 2, given by (4. 1) and (4. 1. 1) can be written:

$${}_jR_k(t), = {}_2R_k(0) + tE_j, \quad j = 1, 2 \tag{9.1}$$

where

$$E_1 = \begin{bmatrix} & & 0 \\ & 0 & \vdots \\ & & 0 \\ 0 & \cdots 0 & 1 \end{bmatrix} \qquad E_2 = \begin{bmatrix} & & r_{1k} \\ & 0 & \\ & & r_{k-1,k} \\ r_{k1} & \cdots \quad r_{k,k-1} & 0 \end{bmatrix}.$$

The equations for $\underline{x} = \underline{x}(t, hI_1)$ and $\underline{\lambda} = \underline{\lambda}(t, hI_1)$ are, from (2. 2) and (2. 3)

$$\begin{bmatrix} R(t) & A' \\ A & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ -\underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{b} \end{bmatrix} \tag{9.2}$$

where A, which implicitly depends on I_1, is always assumed to be of full row rank. Letting $R(t) = {}_jR_k(t)$, it follows, from (9. 1) that

$$t \begin{bmatrix} E_j & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ -\underline{\lambda} \end{bmatrix} + \begin{bmatrix} {}_jR_k(0) & A' \\ A & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ -\underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{b} \end{bmatrix} \tag{9.3}$$

and, hence,

$$\begin{bmatrix} R & A' \\ A & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ -\underline{\lambda} \end{bmatrix} = \left\{ \begin{bmatrix} \underline{0} \\ \underline{b} \end{bmatrix} - t \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ -\underline{\lambda} \end{bmatrix} \right\} = \begin{bmatrix} -tE\underline{x} \\ \underline{b} \end{bmatrix} \tag{9.4}$$

where, to simplify the notation, $R = {}_jR_k(0)$ and $E = E_j$. Since t appears linearly in, at most, on row and one column of $R(t)$, it follows that the coordinates of $\underline{x}$ and $\underline{\lambda}$ will be rational in t, of degree two at most. Therefore, they can be represented by:

$$\underline{x} = \frac{1}{y} \underline{X} , \qquad \underline{\lambda} = \frac{1}{y} \underline{\Lambda} , \tag{9.5}$$

where

$$\begin{aligned} \underline{X} &= \underline{X}_2 t^2 + \underline{X}_1 t + \underline{X}_o , \\ \underline{\Lambda} &= \underline{\Lambda}_2 t^2 + \underline{\Lambda}_1 t + \underline{\Lambda}_o , \\ y &= y_2 t^2 + y_1 t + y_o . \end{aligned} \tag{9.6}$$

Then (9. 4) becomes

$$\begin{bmatrix} R & A' \\ A & 0 \end{bmatrix} \begin{bmatrix} \underline{X} \\ -\underline{\Lambda} \end{bmatrix} = \begin{bmatrix} -tE\underline{X} \\ y\underline{b} \end{bmatrix} . \tag{9.7}$$

By hypothesis, $\begin{bmatrix} R & A' \\ A & O \end{bmatrix}$ is nonsingular, hence, (9. 7) is equivalent to:

$$\begin{bmatrix} \underline{X} \\ -\underline{\Lambda} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} -tE\underline{X} \\ y\underline{b} \end{bmatrix} . \tag{9.8}$$

Equations (9. 7) and (9. 8) are used to determine the unknown coefficients in (9. 6).

As a solution to (9. 3) exists for $t = 0$, it follows that $y_o \neq 0$; hence, we may set $y_o = 1$ since the coefficients in (9. 6) are only specified up to a multiplicative constant. Since (9. 7) and (9. 8) hold for an infinite set of t values, the coefficients of like powers of t on either side of these equations must be identical. Therefore, from (9. 8),

$$\begin{aligned}
\underline{0} &= Z_{11}E\underline{X}_2 \\
\underline{X}_2 &= y_2 Z_{12}\underline{b} - Z_{11}E\underline{X}_1 \\
\underline{X}_1 &= y_1 Z_{12}\underline{b} - Z_{11}E\underline{X}_o \\
\underline{X}_o &= Z_{12}\underline{b} \\
\underline{0} &= Z_{21}E\underline{X}_2 \\
-\underline{\Lambda}_2 &= y_2 Z_{22}\underline{b} - Z_{21}E\underline{X}_1 \\
-\underline{\Lambda}_2 &= y_1 Z_{22}\underline{b} - Z_{21}E\underline{X}_o \\
-\underline{\Lambda}_o &= Z_{22}\underline{b}
\end{aligned} \tag{9.9}$$

and, from (9. 7),

$$A\underline{X}_2 = \underline{b}y_2, \quad A\underline{X}_1 = \underline{b}y_1 . \tag{9.10}$$

The remaining equalities from (9.7) are not needed. Solving (9.9) in terms of y_1 and y_2 yields

$$
\begin{aligned}
\underline{X}_o &= Z_{12}\underline{b} \\
\underline{X}_1 &= y_1 Z_{12}\underline{b} - Z_{11}EZ_{12}\underline{b} \\
\underline{X}_2 &= y_2 Z_{12}\underline{b} - y_1 Z_{11}EZ_{12}\underline{b} + Z_{11}EZ_{11}EZ_{12}\underline{b} \\
\underline{\Lambda}_o &= -Z_{22}\underline{b} \\
\underline{\Lambda}_1 &= -y_1 Z_{22}\underline{b} + Z_{21}EZ_{12}\underline{b} \\
\underline{\Lambda}_2 &= -y_2 Z_{22}\underline{b} + y_1 Z_{21}EZ_{12}\underline{b} - Z_{21}EZ_{11}EZ_{12}\underline{b} \\
\underline{0} &= E(y_2 Z_{12}\underline{b} - y_1 Z_{11}EZ_{12}\underline{b} + Z_{11}EZ_{11}EZ_{12}\underline{b}).
\end{aligned}
\tag{9.11}
$$

The last equation is of rank 0, 1, or 2. If 0, set $y_2 = y_1 = 0$; if 1 set $y_2 = 0$ and solve for y_1 (setting $y_1 = 0$ if its coefficient is 0); if 2 solve for y_1 and y_2. Making use of (9.10), a solution of (9.11) is obtained in each case. Since any solution of (9.11) is a solution of (9.8) for all t, and (9.8) and (9.2) are equivalent systems, it follows that a solution for (9.2) is obtained. This solution must be unique, since the matrix on the left side of (9.2) is nonsingular.

Each time there is a transition or a degeneracy

$$
\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} R_{kj}(0) & A' \\ A & 0 \end{bmatrix}^{-1}
\tag{9.12}
$$

changes, and these changes are of the following types.

(1) Add or subtract one or more rows from A, corresponding to transitions in which Props. 1 or 2 are satisfied.

(2) Border $R_{kj}(0)$ with a row and column and add a column to A in the same position as the column added to $R_{kj}(0)$. This occurs when we add a new variable either to start a new stage or to resolve a degeneracy.

(3) The reverse of (2) occurs when we delete a variable previously added to resolve a degeneracy.

(4) Replace the zeros in the kj'th and jk'th positions of $R_{kj}(0)$ by r_{kj} and $r_{jk}(=r_{kj})$, respectively. This occurs when we pass from one subproblem to the next, after the first in the k'th stage.

All these changes can be accomplished by modifying (9. 12) by well-known methods rather than by reinverting each time.

10. EXTENSION TO LINEAR PROGRAMMING

The extension is obtained by embedding the linear objective function in a second-degree polynomial with a positive definite quadratic component, solving the resulting problem by the algorithm just given and then driving the quadratic part to zero in the final stage. The algorithm is unchanged but some changes are required in the solution for $\underline{x}$ and $\underline{\lambda}$.

If the original linear programming problem is: minimize $\underline{u}'\underline{x}$ subject to $\underline{a}_i{}'\underline{x} \geq b_i$, $i = 1, \ldots, p$, where $\underline{u}$, $\underline{a}_i$, and b_i are all given, replace this problem with the derived problem: minimize $\underline{x}'\underline{x} + \underline{u}'\underline{x}$ subject to the same conditions. Condition (2. 2) is replaced by $R(t)\underline{x} - A'\underline{\lambda} = \underline{u}$; (9. 4), (9. 7) and (9. 8) are replaced by $-tE\underline{x}$ by $\underline{u} - tE\underline{x}$; and in (9. 9) $\underline{X}_o = Z_{12}\underline{b}$ and $-\underline{\Lambda}_o = Z_{22}\underline{b}$ are replaced by, respectively, $\underline{X}_o = Z_{11}\underline{u} + Z_{12}\underline{b}$ and $-\underline{\Lambda}_o = Z_{21}\underline{u} + Z_{22}\underline{b}$. The required changes in (9. 11) are then easily made.

After the solution to this problem has been found, drive $\underline{x}'\underline{x}$ to zero by replacing $\underline{x}'\underline{x} + \underline{u}'\underline{x}$ by $\underline{x}'\underline{x} + t\underline{u}'\underline{x}$ and letting t go from 1 to ∞. (9. 4) is now replaced by

$$\begin{bmatrix} I & A' \\ A & 0 \end{bmatrix} \begin{bmatrix} \underline{x} \\ -\underline{\lambda} \end{bmatrix} = \begin{bmatrix} \underline{u}t \\ \underline{b} \end{bmatrix} \tag{10.1}$$

where $E = 0$. Then, simply,

$$\begin{bmatrix} \underline{x} \\ -\underline{\lambda} \end{bmatrix} = \begin{bmatrix} I & A' \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} \underline{u}t \\ \underline{b} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \underline{u}t \\ \underline{b} \end{bmatrix}. \tag{10.2}$$

If there are n variables then there are at least $n + 1$ subproblems, but, at most, only a few more than $n + 1$: n for the solution of the derived problem, 1 for the elimination of the quadratic component, and possibly a few more due to degeneracies.

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